Interview Questions

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Note

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The purpose of this document is to get students to be comfortable with questions that stray from the routine maths done in school, and ultimately, to prepare for interviews that test your mathematical problem solving.

The questions have been laid out in no particular order and may vary in difficulty.

The author may be contacted on discord (flour#7326) if there are any errors found in this document.

Questions

1. For any positive integer n, prove that $n^3 - n$ is divisible by 6.

2. Sketch the graphs of $\sin(x^2)$ and $\sin^2(x)$. What shape is the first graph near the origin?

3. An upside down cone of height h is filled half way up. What proportion of the volume of the cone is empty?

4. (a) Define $\cosh x$ and $\sinh x$. Sketch both on the same set of axes. Where do both graphs cut the axes? What is the gradient of $\sinh x$ at $x = 0$?
   (b) Show that $\operatorname{arsinh} x = \ln\left(x + \sqrt{1 + x^2}\right)$.

5. Use the formula for a geometric progression to prove that $0.999\cdots = 1$.

6. Let $f(x) = x^3 - 13x^2 + 39x - 27$. Given that the roots of $f(x)$ are in a geometric progression, find the roots of $f(x) = 0$.

7. Analog watch shows that it is midnight and both hands are overlapping. When is the next time they overlap?

8. Sketch the graph of $y = e^{\frac{1}{x}}$.

9. Given that $a \neq b$. How many roots does $(x-a)^2 + (x-b)^2 = 0$ have?

10. How many corners and sides does a square have in n dimensions?

11. Given that ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$.

    What is ${}^{n+1}C_k$ in terms of ${}^nC_k$?

12. For what value of k does $e^x = kx$ have only one solution?

13. For any odd positive integer n, prove that $n^2 - 1$ is divisible by 8.

14. What are the coordinates of the vertices of an octagon with sides of unit length with a centre at the origin?

15. Simplify the following expression

$$\frac{1}{1 + \frac{\sin 2x}{1+\cos 2x}}.$$

16. Which of the following, if any, are prime for $n > 1$?
    A. (n^2-1) B. $(10n^2-1)$ C. $(5n^2-1)$ D. $(8n^2-1)$ E. $(2n^2-1)$

17. A function is bounded if the set of its values is bounded (i.e. the values do not go to infinity). State whether the following function is bounded or not:

$$\int (\sin x)^{1024} e^x \, \mathrm{d}x$$

18. What is the maximum area of an isosceles triangle with two sides of unit length?

19. Sketch the graph of $|y+a| = |x+b|$.

20. Show that $\csc\theta - \cot\theta = \tan\frac{1}{2}\theta$.

21. Sketch the curve $y = x^n e^{-x}$ for when:

    (a) n is even

    (b) n is odd.

22. Evaluate the following integrals:

    (a) $\displaystyle\int_0^n x - \lceil x \rceil \, \mathrm{d}x$ (where $\lceil x \rceil$ is the greatest integer which does not exceed x)

    (b) $\displaystyle\int_0^{n\pi} |\sin x| \, \mathrm{d}x$

    (c) $\displaystyle\int_0^{2\pi} |\cos x - 1| \, \mathrm{d}x$

23. Show that the product of four consecutive integers is 1 less than a perfect square.

24. (a) Show that
$$\left(\frac{x+y}{2}\right)^2 - \left(\frac{x-y}{2}\right)^2 = xy$$
(b) Hence, show that
$$\frac{x+y}{2} \geq \sqrt{xy}.$$

25. Compute $\displaystyle\int_0^{\pi} (x \sin x)^2 \, \mathrm{d}x$.

26. There is a pile of 129 coins on a table. 128 of them are unbiased and the remaining coin has heads on both sides. David chooses a coin at random and tosses it eight times. The coin comes up heads every time. What is the probability that it will come up heads the ninth time as well?

27. Sketch $y = \dfrac{1}{(1-x)^2}$.

28. Twenty balls are placed in an urn. Five are red, five green, five yellow and five blue. Three balls are drawn from the urn at random without replacement. What are the probabilities of the following events:

    (a) Exactly one of the balls drawn is red;
    (b) The three balls drawn have different colours;
    (c) The number of blue balls drawn is strictly greater than the number of yellow balls drawn.

29. In the town of Rejectbridge, 2/3 of the adult men are married to 3/5 of the adult women. Given that each man is married to one woman, what fraction of the adult population of Rejectbridge is married?

30. Is there a two-digit number "ab" such that the difference between "ab" and its reverse "ba" is prime?

31. Given that a and b are distinct real numbers, show that

    (a) $(x-a)^2 + (x-b)^2 = 0$ has no real roots
    (b) $(x-a)^3 + (x-b)^3 = 0$ has one real root.

32. State two numbers, neither ending in zero, which have a product of 10^6.

33. Suppose an unbiased six-sided die is rolled 6 times. What is the probability that the rolls are strictly increasing?

34. Given 5 points on a sphere, can we split the sphere into two hemispheres such that 4 points lie on one of them?

35. An unbiased six-sided die is rolled n times. What is the probability that the die has landed on each face at least once?

36. Given an 8×8 chess board and a set of 31 dominos, such that each domino covers exactly two adjacent squares on the chess board. Two diametrically and diagonally opposite squares are cut away from the chess board (so that it has a total of 62 squares left). Your task is to cover the remaining chessboard with dominos, such that it is completely covered and no dominos overflow from the sides of the modified chess board.

37. (a) By considering the recurring decimal $0.444\ldots$ as the sum of an infinite geometric series express $0.444\ldots$ in the form $\frac{p}{q}$ for any integers p, q.

    (b) Find the sum of the first n terms of the series

$$4 + 44 + 444 + \ldots.$$

38. In music theory, a circle of fifths is constructed by starting at C, and listing all the notes upon moving up seven semitones from the previous. The circle of fifths contains each note exactly once. (Note there are 12 notes in an octave).

    We define an n-circle to be constructed in a fashion similar to that of the circle of fifths, but going up in n semitones. We call such a circle *malformed* if it does not contain every note exactly once, and *well-formed* if it does.

    (a) Prove that the circle of fifths is well-formed.

    (b) Can you define the n-circle for $n > 12$ in terms of a $k - circle$ where $k > 12$?

    (c) Can you find necessary and sufficient conditions for an n-circle to be well-formed?

    We define a k-keyboard to be a keyboard that has k notes in an octave.

    (d) What is a necessary and sufficient condition on k for every n-circle to be malformed?

    (e) Does there exist a positive k such that every n-circle is well-formed?

    Communicated by discord user Waffloid#3141

39. (a) Find the number if ways in which a total of 10 may be obtained by throwing 3 unbiased six-sided dice.

    (b) In a game 4 players roll in turn an unbiased six-sided die until six is obtained. The winner is the first to roll a six. Find the probability that

    (i) the first player wins

    (ii) the last player wins.

40. Ten people sit around a round table. A sum of £10 is to be distributed among them according to the rule that each person receives one half of the sum that his two neighbours receive jointly. Is there only one way to distribute the money?

41. Oliver's stamp collection consists of three books. Two tenths of his stamps are in the first book, several sevenths in the second book and there are 303 stamps in the third book. How many stamps does Oliver have in total?

42. Bob has 10 pockets and 44 coins.

    (a) Can Bob distribute the coins into his pockets such that each pocket contains a different number of coins? (A pocket can have 0 coins)

    (b) Generalise this problem, considering p pockets and n coins.
    Why is the problem most interesting when

$$n = \frac{(p+1)(p-2)}{2}\ ?$$

43. A zero r of $P(x)$ has multiplicity m if $(x-r)$ appears m times in the factorisation of $P(x)$. For example, $x=1$ is a zero of multiplicity 2 of the polynomial x^2-2x+1.

    Prove that if the polynomial $P(x)$ and its derivative $P'(x)$ share the zero $x=r$, then $x=r$ is zero of multiplicity greater than 1.

44. Find the minimum value of

$$\frac{x^2+y^2+1}{x^2+y^2}.$$

45. Evaluate $\displaystyle\int \ln x \ \mathrm{d}x$.

46. (a) Sketch the graphs of:

    (i) $|x|^2+|y|^2=1$

    (ii) $|x|+|y|=1$

    (iii) $|x|^5+|y|^5=1$

    (b) Hence sketch the graph of $|x|^{100}+|y|^{100}=1$.

47. Lockers in a row are numbered $1,2,3,\ldots,100$. At first, all the lockers are closed. A person walks by and opens every other locker, starting with locker #2. Thus lockers $2,4,6,\ldots,98,100$ are open. Another person walks by, and changes the "state" (i.e., closes a locker if it is open, opens a locker if it is closed) of every third locker, starting with locker #3. Then another person changes the state of every fourth locker, starting with #4, etc. How many lockers are closed when the 100th person has passed through?

48. Sketch $y=\dfrac{x^2+1}{x+1}$.

49. (a) Differentiate $y=x^x$.

    (b) Sketch the graph of $y=x^x$ for $x>0$.

    (c) Sketch the graph of $y=x^{\frac{1}{x}}$ for $x>0$.

50. Evaluate $\lim\limits_{x\to 0} x^x$.

51. There are b black chess pieces and w white chess pieces in a bag. A person takes out one chess piece at a time. What is the probability that the last chess piece is white?

52. Sketch the graph of
$$y = \frac{(x-3)(x-2)}{(x+2)(x-1)}.$$

53. There are four cards on the desk. Two are facing up and two are facing down. The two cards facing up are a queen and a nine. The two facing down have a red back and blue back. How many cards need to be turned over to be certain that all cards with red backs are pictures (i.e. kings, queens, jacks)?

54. Prove that if the sum of the digits of a number n are divisible by 3 then n is divisible by 3.

55. Two players, A and B, play game where a biased coin is flipped with a probability p of coming up heads. A player wins if they get heads. If player A gets tails then it is player B's turn to flip the coin and if player B gets tails then it is again player A's turn.
Player A starts the game. Find the probability that player A wins.

56. Find $\frac{\mathrm{d}}{\mathrm{d}x} x^{x^{x^{\cdot^{\cdot^{\cdot}}}}}$

57. Given that ${}^nC_r = \binom{n}{r} = \frac{n!}{r!(n-r)!}$.

Show that ${}^nC_0 +^n C_1 +^n C_2 + \cdots +^n C_n = 2^n$

58. Find the number of common positive divisors of 10^{40} and 20^{30}.

59. A function f is defined for all positive integers. Given that $f(1) = 2015$ and $f(1) + f(2) + f(3) + \cdots + f(n) = n^2 f(n)$ for $n > 1$, calculate $f(2015)$.

60. What is the first positive integer which, when squared, ends in three 4s?

61. (a) Evaluate the following integrals:
    (i) $\int_{-\pi}^{\pi} |\sin x| \, \mathrm{d}x$
    (ii) $\int_{-\pi}^{\pi} \sin |x| \, \mathrm{d}x$
    (iii) $\int_{-\pi}^{\pi} x \sin x \, \mathrm{d}x$
    (iv) $\int_{-\pi}^{\pi} x^{10} \sin x \, \mathrm{d}x$

    (b) What can you say about
$$\int_{-\pi}^{\pi} x^n \sin x \, \mathrm{d}x$$
    for different integer values of n?

62. Compute $\int_{-\frac{1}{2}}^{\frac{1}{2}} \cos x \ln\left(\frac{1+x}{1-x}\right) \mathrm{d}x$.

63. m distinct boys and n distinct girls are arranged in a single row. Find the number of arrangements when:

    (a) there are no restrictions

    (b) no boy is adjacent to another boy

    (c) the n girls form a single block

    (d) a particular boy and a particular girl must be adjacent.

64. How many divisors will n have if $n = p^a q^b$.

65. Find all positive pairs of integers for which $(u+v)^2 = u^3 + v^3$.

66. Find $\arctan x + \arctan\left(\frac{1}{x}\right)$ for any real number x.

67. Let L_1 and L_2 be two lines in the plane with equations $y = m_1x + c_1$ and $y = m_2x + c_2$ respectively. Suppose that they intersect at an acute angle θ. Show that

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|.$$

68. Every subset of the set $\{x_1, x_2, x_3, \ldots, x_n\}$ either contains the element x_i or doesn't (i can be any integer from 1 to n).By making use of combinatorial reasoning, show that:

    (a) $\binom{n-1}{r-1} + \binom{n-1}{r} = \binom{n}{r}$

    (b) $\binom{n-2}{r-2} + 2\binom{n-2}{r-1} + \binom{n-2}{r} = \binom{n}{r}$

69. Prove the following inequalities:

    (a) $x^2 + y^2 \geq 2xy$, for any real numbers x, y.

    (b) $p^4 + q^4 + r^4 + s^4 \geq 4pqrs$, for any real numbers p, q, r, s.

70. There are n people at an event. If A is friends with B then B is friends with A. Is it true that at least two people have the same amount of friends?

71. Does there exist a non-constant polynomial with integer coefficients such that it is prime for every input?

72. Take five points in an equilateral triangle of side length 1. Prove that there is a pair of points which are separated by a distance $d \leq 1/2$.

73. There are nine points in a square of side length 2. Do any three of the points in the square necessarily form a triangle of an area $a < 1/2$?

74. A $2 \times n$ grid is covered with 2×1 dominoes. How many ways are there to do this? What about a $3 \times n$ grid covered with 3×1 dominoes?

75. Solve

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 1$$

for integers x, y, z.

76. An ant starts to crawl along a taut rubber rope 1 km long at a speed of 1 cm per second (relative to the rubber it is crawling on). At the same time, the rope starts to stretch uniformly at a constant rate of 1 km per second, so that after 1 second it is 2 km long, after 2 seconds it is 3 km long, etc. Will the ant ever reach the end of the rope?

77. There are n ants on a rope of length 10m, facing either end of the rope. They all move in the direction that they face at a speed of 10 m/s. If two ants bump into each other on the rope, they start moving in the opposite direction at the same speed. If an ant reaches either end of the rope then it falls off, prove that they will all fall off eventually. Can you give an upper bound on the amount of time until all the ants fall off the rope?

78. There are 50 ants on a 10m line. The 25 left-most ants are moving to the right, and the 25 right-most ants are moving to the left. If two ants bump into each other on the rope, they start moving in the opposite direction at the same speed. If an ant reaches either end of the line then it falls off. How many collisions will there have been in total once all ants have fallen off the end of the line?

79. There are n coins $c_1, c_2, c_3, \ldots, c_n$. Any coin c_k is biased such that the probability of it coming up heads is $\dfrac{1}{2k+1}$. If all n coins are tossed, what is the probability that the number of times heads has come up is odd?

80. Sketch $x^y = y^x$.

81. Is there a sequence of $a_1, a_2, a_3, \ldots$ such that $a_1^m + a_2^m + a_3^m + \cdots = m$ for any positive integer m?

82. Find a positive integer a such that $n^4 + a$ is not prime for any integer n.

83. Prove that out of any seven real numbers, there are two numbers a, b such that

$$0 < \frac{a-b}{1+ab} < \sqrt{3}.$$

84. What is the expected number of times a six-sided die is rolled such that it lands on all faces at least once? What about an n-sided die?

85. You are presented with 12 identically looking balls of which, you are told, only one differs in weight than the others. Find which one it is and whether it is lighter or heavier than the rest. Your only tool is a scales, which can be used by putting balls (and only balls) on either side and observing the outcome. The scales can be used at most three times.

86. A man has 1000 bottles of soda. He finds out that 1 bottle is poisoned. Fortunately, he has 10 mice which he can use to check which bottle is tainted. However, it takes a day to find the result of the test (i.e. if a mouse drinks the poisoned soda, it dies in 24 hrs). The man is throwing a big party the next night. If he runs optimal tests with his mice, how many of the 1000 bottles can he served at the party? e.g. He could have each mouse drink from a bottle. If they all live, he knows that 10 bottles are safe. All mice have to be assigned to bottles right away, because there's only 24 hours to the party.

87. (a) Prove the inequalities
    (i) $e^x \geq x + 1$ for all real numbers x
    (ii) $x - 1 \geq \ln x$ for $x > 0$.
  (b) If the sum of the positive numbers a, b, c is 3, find the range of the possible values of $(a^2 + b^2 + c^2)$.

88. We define $f^n(x) = f(f(f(\ldots f(x)\ldots)))$, where there are n $f()$s.
  (a) Differentiate $f(f(f(x)))$
  (b) Differentiate $f^n(x)$
  (c) Find a sufficient condition for $f^n(k)$ such that it converges as $n \to \infty$
  (d) Sketch the graph of $f^n(x)$ as $n \to \infty$ for $f(x) = \sin(x)$
  (e) What about $f(x) = \cos(x)$?

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89. (a) If a_n denotes the sum of the first n terms of the series in with rth term $\dfrac{(r^2 + r - 1)}{(r^2 + r)}$.
  Show that a_n lies between $(n - 1)$ and n.
  (b) Find the sum of the infinite series with nth term $\dfrac{(n-1)^3}{n!}$.

90. Two identical cylinders of unit radius have axes which intersect at right angles, so that the intersection of the cylinders has four identical curved surfaces. Find the volume of this intersection without the use of calculus.

91. Given that x is not a multiple of π and n is a positive integer, show that

$$\sin x + \sin 3x + \cdots + \sin\left((2n-1)\,x\right) = \sin^2(nx)\csc x.$$

92. Prove that no number in the sequence

$$11, 111, 1111, 11111, \ldots$$

is the square of an integer.

93. Prove that $(n^{n-1} - 1)$ is divisible by $(n-1)^2$ for $n > 1, n \in \mathbb{Z}$.

94. On the same set of axes:

    (a) sketch:

        i. $y = \ln x$
        ii. $y = \log_{10} x$
        iii. $y = \log_2 x$

        Why do they look the way they do?

    (b) Now sketch $y = \log_x e$ on a new set of axes and reason what you do.

    (c) Hence, compute $\lim_{x\to\infty}\left(1+\frac{1}{n}\right)^n$.

95. Prove that the only solution of the equation

$$x^2 + y^2 + z^2 = 2xyz$$

is $x = y = z = 0$ for integers x, y, z.

96. Tom, Dick and Harry travel together. Dick and Harry are good hikers; each walks p miles per hour. Tom has a bad foot and drives a small car in which two people can ride, but not three; the car covers c miles per hour. The three friends adopted the following scheme:

    They start together, Harry rides in the car with Tom, Dick walks. After a while, Tom drops Harry who walks on; Tom returns to pick up Dick and then Tom and Dick ride in the car till they overtake Harry. At this point they change and Harry rides with Tom while Dick walks just as they started. The whole procedure is repeated as often as necessary.

    (a) How much distance do they cover per hour?

    (b) For what fraction of the time taken to travel the journey does the car carry only one person?

    (c) Check the extreme cases $p = 0$ and $p = c$.

97. Sketch $y = x^3 + Ax^2 + B$ for:

    (a) $A > 0, B > 0$

    (b) $A < 0, B > 0$

98. Jack and Sam live at opposite ends of the same street. Jack had to deliver a parcel to Sam's house and Sam had to deliver one to Jack's house. They started at the same moment, each walked a constant speed and returned home immediately after leaving the parcel at its destination. They met first time at the distance of a metres from Jack's house and the second time b metres from Sam's house.

    (a) How long is the street?

    (b) If $a = 300$ and $b = 400$, who walks faster?

99. Place eight rooks on a standard eight by eight chessboard so that no two are in the same row or column. (Non-attacking rooks.) Now color 27 of the remaining squares red (squares not currently occupied by rooks). Prove that it is always possible to move the rooks to a different set of eight uncolored squares (i.e., at least one formerly empty square now has a rook on it) so that they are still non-attacking. Show that this is not necessarily the case if we are allowed to color 28 squares red.

100. For which value(s) of k is $\binom{n}{k}$ a maximum when n is a given positive integer? Prove your answer.

101. Prove that 101 is prime, but $10101, 1010101, 101010101, \ldots$ are not.

102. (a) A hexagon has sides $a_1, a_2, a_3, \ldots, a_6$ with a circle inscribed inside that touches each side of the hexagon. Given that the hexagon is not necessarily regular, prove that the sum of alternate sides is equal i.e. $a_1 + a_3 + a_5 = a_2 + a_4 + a_6$.

    (b) Now prove this for all even sided shapes.

103. How many digits does 100! have?

104. (a) Sketch

        (i) $y = \ln(\sin x)$ for $0 < x < \pi$

        (ii) $y = \ln(\cos x)$ for $0 < x < \frac{\pi}{2}$

    (b) Evaluate $\int_0^{\frac{\pi}{2}} \ln(\sin x)\,\mathrm{d}x$

105. You have two identical crystal orbs. Your goal is to figure out how high a floor an orb can fall from a 100 story building before it breaks. What is the largest number of orb-drops you would ever have to do to find the right floor (i.e. what is the most efficient strategy by a worst-case measure)? You can break both orbs in your search, provided you yield a unique and correct answer.

106. A game is played with two players and an initial stack of n pennies ($n \geq 3$). The players take turns choosing one of the stacks of pennies on the table and splitting it into two stacks. When a player makes a move that causes all the stacks to be of height 1 or 2 at the end of his or her turn, that player wins. Which starting values of n are wins for each player?

107. Suppose there is an invisible rabbit which is moving along a straight line from an unknown initial position and it jumps and covers a fixed horizontal distance every second. Every second, before it jumps, you can pick a point on the train track to try and catch the rabbit. Can the rabbit be caught in a finite amount of time?

108. X and Y are two different whole numbers greater than 1. Their sum is not greater than 100, and Y is greater than X. S and P are two mathematicians (and consequently perfect logicians); S knows the sum $X + Y$ and P knows the product $X \times Y$. Both S and P know all the information in this paragraph.

The following conversation occurs (both participants are telling the truth):

- S says "P does not know X and Y."
- P says "Now I know X and Y."
- S says "Now I also know X and Y."

What are X and Y?