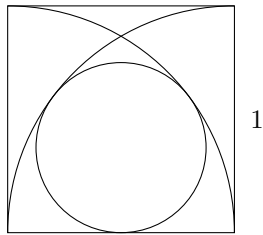


Maths Problems Set 4 (24 March 2020)

1. 5 points are placed in a 1×1 square. Show that no matter where the points are placed there will always be a pair of points with a distance of less than or equal to $\frac{1}{\sqrt{2}}$ between them.
2. I have two identical but non-uniform ropes. I know that if I light one end of either rope it will burn for exactly one hour. How can I time 45 minutes? (I have a lighter of some sort).
3. The minute and hour hands on a analog clock overlap at 12 o'clock. Where else do they overlap?
4. A positive integer ends in the digit 4 and has the property that it becomes four times as large when the 4 is moved from the end and placed at the front. What is the smallest such number?
5. Two quarter circles are inscribed in a square of side length 1 as shown in the diagram below. A circle is then inscribed as shown. Find the radius of the circle.



6. A triangle T has side lengths a , b , and c where c is the longest side length. Prove that: T is right-angled $\iff a^2 + b^2 = c^2$.
7. Newton's law of cooling states that the rate of heat transfer of a body is directly proportional to the difference between the temperature of the body and that of its surroundings. This leads the objects to tend towards the same temperature. A can of Coke with temperature 15 degrees is placed in a fridge kept at constant temperature of 5 degrees. Given that it takes 2 minutes for the can of coke to reach 10 degrees how long does it take to reach a temperature of 7.5 degrees?
8. Siddharth and Dan are playing a tennis match. For each point there is a probability of p that Siddharth wins. What is the probability that Siddharth wins the game?
9. You are walking along a route in a city with square blocks. You need to travel six blocks North and and six blocks East. The shortest possible route is therefore twelve blocks. How many different such twelve-block routes exist?

10. Let $ABCD$ be a quadrilateral. Let A' be the midpoint of AB , B' the midpoint of BC , C' the midpoint of CD , and D' the midpoint of AD . Draw the lines $A'C'$ and $B'D'$, and let their point of intersection be point M . Let a be the area of quadrilateral $D'MA'A$, b the area of quadrilateral $A'MB'B$, c the area of quadrilateral $B'MC'C$, and d the area of quadrilateral $C'MD'D$. Prove that $a + c = b + d$.
11. In Secret Santa, a group of people each pick out the name of a single other person at random, such that no one picks out their own name. In a Secret Santa group of n people, find in terms of n the probability that everyone picks out the same person that picked them.
12. (a) Evaluate $\int_{-\pi}^{\pi} |\sin x + \cos x| \, dx$.
 (b) Find $\int \sec x \, dx$ and $\int \operatorname{cosec} x \, dx$.
 (c) Evaluate $\int_0^1 (1 - \sqrt{x})^{20} \, dx$.
13. Let ABC be a triangle with $\angle CAB$ being a right angle. The point L lies on side BC . A circle passing through points A , B , and L meets the line AC again at a point M and the circle passing through points A , C , and L meets the line AB again at a point N . Prove that L , M , and N are collinear.
14. Jacob the gnome wizard is floating in space in the vicinity of a planet and no other bodies. As Jacob starts hurtling towards the planet, he decides to have one last adventure: to calculate the time until he completes his journey on impact. The mass of the planet is 2×10^{25} kg and the radius of the planet is 7×10^8 m. Jacob's initial height h above the surface of the planet is 5×10^8 m and his initial velocity u is 200 ms^{-1} . What is the time taken to crash into the planet?
15. n points are placed on the circumference of a circle, and each pair of points is joined by a straight line. The points are chosen so that no three of these lines pass through the same point. In terms of n , how many regions is the circle's interior cut into?
16. How many ways are there to select a list of r objects from a list of n possible options where each object can appear multiple times and order doesn't matter? If there is a list of 3 elements: A , B , and C , the possible ways to select 2 items are: AA , AB , AC , BB , BC , CC . Notice that BA is the same as AB , CB is the same as BC , and so on. Therefore, there are 6 ways to select 2 items from 3 possible options with these constraints.